

ON THE CHIRAL WZNW PHASE SPACE, EXCHANGE r -MATRICES AND POISSON-LIE GROUPOIDS*

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ABSTRACT. This is a review of recent work on the chiral extensions of the WZNW phase space describing both the extensions based on fields with generic monodromy as well as those using Bloch waves with diagonal monodromy. The symplectic form on the extended phase space is inverted in both cases and the chiral WZNW fields are found to satisfy quadratic Poisson bracket relations characterized by monodromy dependent exchange r -matrices. Explicit expressions for the exchange r -matrices in terms of the arbitrary monodromy dependent 2-form appearing in the chiral WZNW symplectic form are given. The exchange r -matrices in the general case are shown to satisfy a new dynamical generalization of the classical modified Yang-Baxter (YB) equation and Poisson-Lie (PL) groupoids are constructed that encode this equation analogously as PL groups encode the classical YB equation. For an arbitrary simple Lie group G , exchange r -matrices are exhibited that are in one-to-one correspondence with the possible PL structures on G and admit them as PL symmetries.

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1 Introduction

The Wess-Zumino-Novikov-Witten (WZNW) model [1] of conformal field theory has proved to be the source of interesting structures that play an increasingly important rôle in theoretical physics and in mathematics [2, 3]. One of the fascinating aspects of the model is that in addition to its built-in affine Kac-Moody symmetry it also exhibits certain quantum group properties [4]. The quantum group properties were originally discovered in the quantized model, which raised the question to find their classical analogues. The studies at the beginning of the nineties led to the consensus that the origin of these quantum group properties lies in the Poisson-Lie symmetries of the so called chiral WZNW phase space that emerges after splitting the left- and right-moving degrees of freedom [5]-[16]. The chiral separation arises from the product form of the solution of the WZNW field equation given by $g(x_L, x_R) = g_L(x_L)g_R^{-1}(x_R)$, where x_C ($C = L, R$) are lightcone coordinates and the g_C are quasiperiodic group valued fields with equal monodromies, $g_C(x + 2\pi) = g_C(x)M$ for some M in the WZNW group G . The chiral WZNW Poisson structures found in the literature have the form

$$\{g_C(x) \otimes g_C(y)\} = \frac{1}{\kappa_C} (g_C(x) \otimes g_C(y)) \left(\hat{r} + \frac{1}{2} \hat{I} \text{sign}(y - x) \right), \quad 0 < x, y < 2\pi, \quad (1.1)$$

where \hat{I} is given by the quadratic Casimir of the simple Lie algebra, \mathcal{G} , of the WZNW group, G , and the interesting object is the ‘exchange r-matrix’ \hat{r} . These classical ‘exchange algebras’ can be regarded as fundamental since the current algebra follows as their consequence, and help to better understand the quantum group properties of the model by means of canonical quantization [17, 18, 19]. However, the choice of the chiral Poisson structure is highly non-unique due to the fact that the g_C are determined by the physical field g only up to the gauge freedom $g_C \mapsto g_C h$ for any constant $h \in G$.

There are two qualitatively different cases that correspond to building the WZNW field out of chiral fields with diagonal monodromy (‘Bloch waves’) or out of fields with generic monodromy. For Bloch waves [9, 10, 11], the Poisson structure is essentially unique and the associated r-matrix is a solution of the so called classical dynamical Yang-Baxter (CDYB) equation, which has recently received a lot of attention [20]. For chiral fields with generic monodromy, it has been argued in [12, 15] that the possible exchange r-matrices should correspond to certain local differential 2-forms ρ on open domains $\tilde{G} \subset G$, whose exterior derivative is the 3-form that occurs in the WZNW action. Until recently, the precise connection between ρ and \hat{r} has not been elaborated, and in most papers dealing with generic monodromy actually only those very special cases were considered for which \hat{r} is a monodromy independent constant.

We here review the main results obtained in our recent papers [21, 22], where a detailed analysis of the chiral extensions of the WZNW phase space was undertaken. The next section contains an outline of the background to the problem. The subsequent

two sections describe the chiral WZNW hamiltonian structures in detail. The final section is devoted to an interpretation of these structures in terms of Poisson-Lie groupoids.

2 Chiral extensions of the WZNW phase space

We below describe the WZNW Hamiltonian system and the chiral extension of its solution space following the spirit of [12, 15].

We consider a simple, real or complex, Lie algebra, \mathcal{G} , with a corresponding connected Lie group, G , and identify the phase space of the WZNW model associated with the group G as

$$\mathcal{M} = T^*\tilde{G} = \{ (g, J_L) \mid g \in \tilde{G}, J_L \in \tilde{\mathcal{G}} \}, \quad (2.1)$$

where $\tilde{G} = C^\infty(S^1, G)$ is the loop group and $\tilde{\mathcal{G}} = C^\infty(S^1, \mathcal{G})$ is its Lie algebra. The isomorphism of the cotangent bundle $T^*\tilde{G}$ with $\tilde{G} \times \tilde{\mathcal{G}}$ is established by means of right-translations on \tilde{G} . The elements $g \in \tilde{G}$ (resp. $J_L \in \tilde{\mathcal{G}}$) are modeled as 2π -periodic G -valued (resp. \mathcal{G} -valued) functions on the real line \mathbf{R} . The phase space is equipped with the symplectic form

$$\Omega^\kappa = d \int_0^{2\pi} d\sigma \operatorname{Tr} (J_L dg g^{-1}) + \frac{\kappa}{2} \int_0^{2\pi} d\sigma \operatorname{Tr} (dg g^{-1}) \wedge (dg g^{-1})' \quad (2.2)$$

with some constant κ . Here prime denotes derivative with respect to the space variable, $\sigma \in \mathbf{R}$, and for any $A, B \in \mathcal{G}$ $\operatorname{Tr}(AB)$ denotes a fixed multiple of the Cartan-Killing form on \mathcal{G} . If T_α and T^α ($\alpha = 1, \dots, \dim \mathcal{G}$) are dual bases of \mathcal{G} , $\operatorname{Tr}(T_\alpha T^\beta) = \delta_\alpha^\beta$, then $\operatorname{Tr}(AB) = A_\alpha B^\alpha$ with $A_\alpha = \operatorname{Tr}(AT_\alpha)$, $B^\alpha = \operatorname{Tr}(BT^\alpha)$ and the usual summation convention in force. For the wedge product we use the conventions in [23].

Although the expression of Ω^κ appears rather formal at first sight, it can be used to unambiguously associate hamiltonian vector fields and Poisson brackets (PBs) with a set of admissible functions, which include, for example, the Fourier components of the WZNW field g , the ‘left-current’ J_L and the ‘right-current’ J_R given by

$$J_R = -g^{-1} J_L g + \kappa g^{-1} g'. \quad (2.3)$$

The currents are the momentum maps that generate two commuting actions of \tilde{G} on \mathcal{M} that correspond respectively to left- and right-translations on \tilde{G} . This means that the following local PB relations are valid:

$$\begin{aligned} \{ \operatorname{Tr}(T_\alpha J_L)(\sigma), \operatorname{Tr}(T_\beta J_L)(\bar{\sigma}) \}_{WZ} &= \operatorname{Tr}([T_\alpha, T_\beta] J_L)(\sigma) \delta + \kappa \operatorname{Tr}(T_\alpha T_\beta) \delta' \\ \{ \operatorname{Tr}(T_\alpha J_R)(\sigma), \operatorname{Tr}(T_\beta J_R)(\bar{\sigma}) \}_{WZ} &= \operatorname{Tr}([T_\alpha, T_\beta] J_R)(\sigma) \delta - \kappa \operatorname{Tr}(T_\alpha T_\beta) \delta' \\ \{ g(\sigma), \operatorname{Tr}(T_\alpha J_L)(\bar{\sigma}) \}_{WZ} &= T_\alpha g(\sigma) \delta \\ \{ g(\sigma), \operatorname{Tr}(T_\alpha J_R)(\bar{\sigma}) \}_{WZ} &= -g(\sigma) T_\alpha \delta, \end{aligned} \quad (2.4)$$

together with $\{J_L(\sigma), J_R(\bar{\sigma})\}_{WZ} = 0$, where $\delta := \delta(\sigma - \bar{\sigma}) = \frac{1}{2\pi} \sum_{n \in \mathbf{Z}} e^{in(\sigma - \bar{\sigma})}$. These PBs can be derived from the symplectic form Ω^κ , whose precise meaning is explained in several papers (see e.g. [24]). Thus we need not dwell on this point, but note that in the case of a complex Lie algebra the admissible functions depend holomorphically on the matrix elements of g , J_L , J_R in the finite dimensional irreducible representations of G , and $\tilde{G} \times \tilde{G}$ is then a model of the holomorphic cotangent bundle.

The phase space \mathcal{M} represents the initial data for the WZNW system, whose dynamics is generated by the Hamiltonian

$$H_{WZ} = \frac{1}{2\kappa} \int_0^{2\pi} d\sigma \operatorname{Tr} (J_L^2 + J_R^2). \quad (2.5)$$

Denoting time by τ and introducing lightcone coordinates as

$$x_L := \sigma + \tau, \quad x_R := \sigma - \tau, \quad \partial_L = \frac{\partial}{\partial x_L} = \frac{1}{2}(\partial_\sigma + \partial_\tau), \quad \partial_R = \frac{\partial}{\partial x_R} = \frac{1}{2}(\partial_\sigma - \partial_\tau), \quad (2.6)$$

Hamilton's equation can be written in the alternative forms [1]

$$\kappa \partial_L g = J_L g, \quad \partial_R J_L = 0 \quad \Leftrightarrow \quad \kappa \partial_R g = g J_R, \quad \partial_L J_R = 0. \quad (2.7)$$

Let \mathcal{M}^{sol} be the space of solutions of the WZNW system. \mathcal{M}^{sol} consists of the smooth G -valued functions $g(\sigma, \tau)$ which are 2π -periodic in σ and satisfy $\partial_R(\partial_L g g^{-1}) = 0$. The general solution of this evolution equation can be written as

$$g(\sigma, \tau) = g_L(x_L) g_R^{-1}(x_R), \quad (2.8)$$

where (g_L, g_R) is any pair of G -valued, smooth, quasiperiodic function on \mathbf{R} with *equal monodromies*, i.e., for $C = L, R$ one has $g_C(x_C + 2\pi) = g_C(x_C)M$ with some C -independent $M \in G$. To elaborate this representation of the solutions in more detail, we define the space $\widehat{\mathcal{M}}$:

$$\widehat{\mathcal{M}} := \{(g_L, g_R) | g_{L,R} \in C^\infty(\mathbf{R}, G), \quad g_{L,R}(x + 2\pi) = g_{L,R}(x)M \quad M \in G\}. \quad (2.9)$$

There is a free right-action of G on $\widehat{\mathcal{M}}$ given by

$$G \ni h : (g_L, g_R) \mapsto (g_L h, g_R h). \quad (2.10)$$

Notice that $\widehat{\mathcal{M}}$ is a *principal fibre bundle* over \mathcal{M}^{sol} with respect to the above action of G . The projection of this bundle, $\vartheta : \widehat{\mathcal{M}} \rightarrow \mathcal{M}^{sol}$, is given by

$$\vartheta : (g_L, g_R) \mapsto g = g_L g_R^{-1} \quad \text{i.e.} \quad g(\sigma, \tau) = g_L(x_L) g_R^{-1}(x_R). \quad (2.11)$$

We can identify \mathcal{M} with \mathcal{M}^{sol} by associating the elements of the solution space with their initial data at $\tau = 0$. Formally, this is described by the map $\iota : \mathcal{M}^{sol} \rightarrow \mathcal{M}$,

$$\iota : \mathcal{M}^{sol} \ni g(\sigma, \tau) \mapsto (g(\sigma, 0), J_L(\sigma) = (\kappa \partial_L g g^{-1})(\sigma, 0)) \in \mathcal{M}. \quad (2.12)$$

Obviously, $\iota^*(\Omega^\kappa)$ is then the natural symplectic form on the solution space. Explicitly,

$$(\iota^*\Omega^\kappa)(g) = -\kappa \left(d \int_0^{2\pi} d\sigma \operatorname{Tr} \left(g^{-1} \partial_R g g^{-1} dg \right) + \frac{1}{2} \int_0^{2\pi} d\sigma \operatorname{Tr} \left(g^{-1} dg \right) \wedge \partial_\sigma \left(g^{-1} dg \right) \right) \Big|_{\tau=0} \quad (2.13)$$

Regarding now \mathcal{M}^{sol} as the base of the bundle $\vartheta : \widehat{\mathcal{M}} \rightarrow \mathcal{M}^{sol}$, we obtain a closed 2-form, $\widehat{\Omega}^\kappa$, on $\widehat{\mathcal{M}}$, $\widehat{\Omega}^\kappa := \vartheta^*(\iota^*\Omega^\kappa)$. By substituting the explicit formula (2.11) of ϑ , one finds

$$\widehat{\Omega}^\kappa(g_L, g_R) = \kappa_L \Omega_{chir}(g_L) + \kappa_R \Omega_{chir}(g_R), \quad \text{with} \quad \kappa_L := \kappa, \quad \kappa_R := -\kappa, \quad (2.14)$$

where Ω_{chir} is the so called chiral WZNW 2-form:

$$\begin{aligned} \Omega_{chir}(g_C) &= -\frac{1}{2} \int_0^{2\pi} dx_C \operatorname{Tr} \left(g_C^{-1} dg_C \right) \wedge \left(g_C^{-1} dg_C \right)' - \frac{1}{2} \operatorname{Tr} \left((g_C^{-1} dg_C)(0) \wedge dM_C M_C^{-1} \right), \\ M_C &= g_C^{-1}(x) g_C(x + 2\pi). \end{aligned} \quad (2.15)$$

This crucial formula of $\widehat{\Omega}^\kappa$ was first obtained by Gawedzki [12].

It is clear from its definition that $d\widehat{\Omega}^\kappa = 0$, but $\widehat{\Omega}^\kappa$ is not a symplectic form on $\widehat{\mathcal{M}}$, since it is degenerate. Of course, its restriction to any (local) section of the bundle $\vartheta : \widehat{\mathcal{M}} \rightarrow \mathcal{M}^{sol}$ is a symplectic form, since such sections yield (local) models of \mathcal{M}^{sol} . On the other hand, one can check that Ω_{chir} has a non-vanishing exterior derivative [12]:

$$d\Omega_{chir}(g_C) = -\frac{1}{6} \operatorname{Tr} \left(M_C^{-1} dM_C \wedge M_C^{-1} dM_C \wedge M_C^{-1} dM_C \right). \quad (2.16)$$

Although this cancels from $d\widehat{\Omega}^\kappa$, since $M_L = M_R$ for the elements of $\widehat{\mathcal{M}}$, it makes the chiral separation of the WZNW degrees of freedom a very non-trivial problem.

The idea of the chiral separation arises from the observation that the currents J_C *almost* completely determine the chiral WZNW fields g_C , and thus also $g = g_L g_R^{-1}$, by means of the differential equations

$$\kappa_C \partial_C g_C = J_C g_C \quad \text{for} \quad C = L, R. \quad (2.17)$$

Thus it appears an interesting possibility to construct the WZNW model as a reduction of a simpler model, in which the left- and right-moving degrees of freedom would be separated in terms of *completely independent* chiral fields g_L and g_R regarded as fundamental variables. It is clear that the solution space of such a chirally extended model must be a direct product of two identical but independent spaces, i.e., it must have the form

$$\widehat{\mathcal{M}}^{ext} := \mathcal{M}_L \times \mathcal{M}_R, \quad (2.18)$$

$$\mathcal{M}_C := \{g_C \mid g_C \in C^\infty(\mathbf{R}, G), \quad g_C(x + 2\pi) = g_C(x) M_C \quad M_C \in G\}. \quad (2.19)$$

Ideally, one would like to endow the space $\widehat{\mathcal{M}}^{ext}$ with a symplectic structure, $\widehat{\Omega}_{ext}^\kappa$, that reduces to $\widehat{\Omega}^\kappa$ on the submanifold $\widehat{\mathcal{M}} \subset \widehat{\mathcal{M}}^{ext}$ defined by the periodicity constraint $M_L = M_R$. It is easy to see that these requirements force $\widehat{\Omega}_{ext}^\kappa$ to have the following form:

$$\widehat{\Omega}_{ext}^\kappa(g_L, g_R) = \kappa_L \Omega_{chir}^\rho(g_L) + \kappa_R \Omega_{chir}^\rho(g_R), \quad (2.20)$$

$$\Omega_{chir}^\rho(g_C) = \Omega_{chir}(g_C) + \rho(M_C) \quad (2.21)$$

with some 2-form ρ depending *only* on the monodromy of g_C . Since in the extended model the factors $(\mathcal{M}_C, \kappa_C \Omega_{chir}^\rho)$ should be *symplectic manifolds separately*, the condition

$$d\Omega_{chir}^\rho = -\frac{1}{6} \text{Tr} \left(M_C^{-1} dM_C \wedge M_C^{-1} dM_C \wedge M_C^{-1} dM_C \right) + d\rho(M_C) = 0 \quad (2.22)$$

arises. But then we have to face the problem that no globally defined smooth 2-form exists on G that would satisfy this condition for all $M_C \in G$.

There are two rather different way-outs from the above difficulty [15]. The first is to restrict the possible domain of the monodromy matrix M_C to some open submanifold in G on which an appropriate 2-form ρ may be found. We refer to a choice of such a domain and 2-form ρ as *a chiral extension of the WZNW system* with generic monodromy.

The second possibility is to restrict the domain of the allowed monodromy matrices much more drastically from the beginning, in such a way that after the restriction $d\Omega_{chir}$ vanishes, whereby the difficulty disappears. For example, one may achieve this by restricting the monodromy matrices to vary in a fixed maximal torus of G , which amounts to constructing (a subset of) the solutions of the WZNW field equation in terms of chiral ‘Bloch waves’. This second possibility is especially natural in the case of compact or complex Lie groups, for which there is only one maximal torus up to conjugation. Geometrically, the restriction to Bloch waves corresponds to taking a (local) section of the bundle $\vartheta : \widehat{\mathcal{M}} \rightarrow \mathcal{M}^{sol}$.

3 Hamiltonian structures for generic monodromy

We here investigate the chiral WZNW phase space \mathcal{M}_C introduced above. The analysis is the same for both chiralities, $C = L, R$, and we simplify our notation by putting \mathcal{M}_{chir} for \mathcal{M}_C and g, M, κ for g_C, M_C, κ_C , respectively. Thus \mathcal{M}_{chir} is parametrized by the G -valued, smooth, quasiperiodic field $g(x)$ satisfying the monodromy condition

$$g(x + 2\pi) = g(x)M \quad M \in G. \quad (3.1)$$

The corresponding chiral current, $J(x) = \kappa g'(x)g^{-1}(x) \in \mathcal{G}$, is a smooth, 2π -periodic function of x . We then consider a 2-form ρ on a domain $\check{G} \subset G$, for which we assume that $d\rho(M) = \frac{1}{6} \text{Tr} (M^{-1} dM)^{3\wedge}$ and let $\check{\mathcal{M}}_{chir} \subset \mathcal{M}_{chir}$ be the set of chiral WZNW fields whose monodromy matrix lies in \check{G} . It turns out that $\kappa \Omega_{chir}^\rho$ defines a symplectic structure on $\check{\mathcal{M}}_{chir}$ if a further condition holds for the pair (\check{G}, ρ) . In order to describe this condition let us introduce the parametrization

$$\rho(M) = \frac{1}{2} q^{\alpha\beta}(M) \text{Tr}(T_\alpha M^{-1} dM) \wedge \text{Tr}(T_\beta M^{-1} dM), \quad q^{\alpha\beta} = -q^{\beta\alpha}, \quad (3.2)$$

with T_α denoting a basis of \mathcal{G} , whose dual basis with respect to Tr is T^α . For any $M \in \check{G}$, then also introduce the linear operator $q(M) : \mathcal{G} \rightarrow \mathcal{G}$ by

$$q(M) : T^\beta \mapsto q^{\alpha\beta}(M)T_\alpha, \quad (3.3)$$

as well as its shifts, $q_\pm(M) := q(M) \pm \frac{1}{2}I$, by the identity operator I . The further condition that we need is that

$$\det \left(q_+(M) - q_-(M) \circ \text{Ad } M^{-1} \right) \neq 0 \quad \forall M \in \check{G}. \quad (3.4)$$

This condition will guarantee the (weak) non-degeneracy of $\kappa\Omega_{chir}^\rho$ on $\check{\mathcal{M}}_{chir}$.

To use $\kappa\Omega_{chir}^\rho$ in practice we need to establish some notation for tangent vectors $X[g]$ at $g \in \mathcal{M}_{chir}$ and vector fields X over the chiral phase space. To this end we consider smooth curves on \mathcal{M}_{chir} described by functions $\gamma(x, t) \in G$ satisfying

$$\gamma(x + 2\pi, t) = \gamma(x, t)M(t) \quad M(t) \in G; \quad \gamma(x, 0) = g(x). \quad (3.5)$$

$X[g]$ is obtained as the velocity to the curve at $t = 0$, encoded by the \mathcal{G} -valued, smooth function

$$\xi(x) := \left. \frac{d}{dt} g^{-1}(x) \gamma(x, t) \right|_{t=0}. \quad (3.6)$$

The monodromy properties of $\xi(x)$ can be derived by taking the derivative of the first equation in (3.5): $\xi'(x + 2\pi) = M^{-1}\xi'(x)M$, and this can be solved in terms of a \mathcal{G} -valued, smooth, 2π -periodic function, $X_J \in \check{\mathcal{G}}$, and a constant Lie algebra element, ξ_0 , as follows:

$$\xi(x) = \xi_0 + \int_0^x dy g^{-1}(y) X_J(y) g(y). \quad (3.7)$$

A vector field X on \mathcal{M}_{chir} is an assignment, $g \mapsto X[g]$, of a vector to every point $g \in \mathcal{M}_{chir}$. Thus it can be specified by the assignments $g \mapsto \xi_0[g] \in \mathcal{G}$ and $g \mapsto X_J[g] \in \check{\mathcal{G}}$. Using any curve that defines $X[g]$, X acts on a differentiable function, $g \mapsto F[g]$, on \mathcal{M}_{chir} as

$$X(F)[g] = \left. \frac{d}{dt} F[g_t] \right|_{t=0} \quad g_t(x) = \gamma(x, t). \quad (3.8)$$

Note that the evaluation functions $F^x[g] := g(x)$ and $\mathcal{F}^x[g] := J(x)$ are differentiable with respect to any vector field, and their derivatives are given by

$$X(g(x)) = g(x)\xi(x) \quad \text{and} \quad X(J(x)) = \kappa X_J(x). \quad (3.9)$$

This clarifies the meaning of X_J as well. It is also obvious from its definition that the monodromy matrix yields a G -valued differentiable function on \mathcal{M}_{chir} , $g \mapsto M = g^{-1}(x)g(x + 2\pi)$, whose derivative is characterized by the \mathcal{G} -valued function

$$X(M)M^{-1} = M\xi(x + 2\pi)M^{-1} - \xi(x). \quad (3.10)$$

Having defined vector fields, one can also introduce differential forms as usual. We only remark that by (3.9) evaluation 1-forms like $dg(x)$, $dJ(x)$ or $(g^{-1}dg)'(x)$ are perfectly well-defined: e.g. $dg(x)(X) = X(g(x)) = g(x)\xi(x)$.

Let us now show that Ω_{chir}^ρ is weakly non-degenerate, that is $\Omega_{chir}^\rho(X, Y) = 0 \ \forall X$ only for $Y = 0$, if and only if (3.4) holds. In order to compute

$$\Omega_{chir}^\rho(X, Y) = \Omega_{chir}(X, Y) + \rho(X, Y) \quad (3.11)$$

for two arbitrary vector fields, we take X to be parametrized by $\xi(x)$ and further by the pair $(\xi_0, X_J(x))$, while the analogous parametrization for Y is given by $\eta(x)$ and the pair $(\eta_0, Y_J(x))$. Then a straightforward calculation gives that

$$\begin{aligned} \Omega_{chir}^\rho(X, Y) &= \int_0^{2\pi} dx \text{Tr} \left(X_J(x) g(x) \left(\eta(x) + q_-(M)(M^{-1}Y(M)) \right) g^{-1}(x) \right) \\ &+ \text{Tr} \left(\xi_0 (q_-(M) - \text{Ad } M \circ q_+(M)) (M^{-1}Y(M)) \right). \end{aligned} \quad (3.12)$$

This vanishes for every X , that is for arbitrary $X_J \in \tilde{\mathcal{G}}$ and $\xi_0 \in \mathcal{G}$, if and only if

$$(q_-(M) - \text{Ad } M \circ q_+(M)) (M^{-1}Y(M)) = 0, \quad \eta(x) + q_-(M)(M^{-1}Y(M)) = 0. \quad (3.13)$$

Since the transpose with respect to the scalar product on \mathcal{G} satisfies

$$(q_-(M) - \text{Ad } M \circ q_+(M))^T = (q_-(M) \circ \text{Ad } M^{-1} - q_+(M)), \quad (3.14)$$

if (3.4) holds then it follows from (3.13) that $\eta(x)$ must vanish, that is $Y = 0$. Thus we proved that (3.4) implies the non-degeneracy of Ω_{chir}^ρ . The converse statement is also easy to establish, since if the determinant in (3.4) vanished say at M^0 , then there would exist a non-zero $A \in \mathcal{G}$ such that $(q_-(M^0) - \text{Ad } M^0 \circ q_+(M^0))(A) = 0$. We could hence define a tangent vector Y^0 at a corresponding point in \mathcal{M}_{chir} by $\eta^0(x) = -q_-(M^0)(A)$, and this vector would annihilate Ω_{chir}^ρ . (The definition of Y^0 is consistent since $\eta^0(x + 2\pi) - (M^0)^{-1}\eta^0(x)M^0 = A$ holds.)

Now we turn to our main problem: For a differentiable (scalar) function F on the phase space $\check{\mathcal{M}}_{chir}$, we wish to find a corresponding vector field, Y^F , satisfying

$$X(F) = \kappa \Omega_{chir}^\rho(X, Y^F) \quad (3.15)$$

for all vector fields X . Notice that Y^F does not necessarily exist for a given F . We say that F is an element of the set of *admissible Hamiltonians*, denoted as \mathcal{H} , if the corresponding hamiltonian vector field, Y^F , exists. On account of the non-degeneracy of Ω_{chir}^ρ , if Y^F exists then it is uniquely determined.

We may use the formula (3.12) for $Y := Y^F$ to establish the following three necessary and sufficient conditions that F must obey to guarantee that Y^F exists:

- There must exist a *smooth* \mathcal{G} -valued function on \mathbf{R} , $A^F(x)$, and a constant Lie algebra element, a^F , such that for any vector field X

$$X(F) = \kappa \int_0^{2\pi} dx \text{Tr} \left(X_J(x) A^F(x) \right) + \kappa \text{Tr} (\xi_0 a^F). \quad (3.16)$$

(This means that $F \in \mathbb{H}$ must have an exterior derivative parametrized by the assignments $g \mapsto A^F(x)[g]$ and $g \mapsto a^F[g]$. The restriction of $A^F(x)$ to $x \in [0, 2\pi]$ and a^F are uniquely determined by (3.16), and $A^F(x)$ is made a unique function on \mathbf{R} by the next requirement.)

- The expression

$$\left[A^F(x), J(x) \right] + \kappa \frac{dA^F(x)}{dx} \quad (3.17)$$

must define a smooth 2π -periodic function on \mathbf{R} .

- $A^F(x)$ and a^F must be related by

$$a^F = g^{-1}(0) \left[A^F(0) - A^F(2\pi) \right] g(0). \quad (3.18)$$

If these conditions are satisfied, then Y^F is in fact given by

$$g^{-1}(x) Y^F(g(x)) = g^{-1}(x) A^F(x) g(x) - \frac{1}{2} a^F + r(M)(a^F), \quad (3.19)$$

where $r(M)$ is the linear operator on \mathcal{G} defined by

$$r(M) = \frac{1}{2} \left(q_+(M) - q_-(M) \circ \text{Ad}(M^{-1}) \right)^{-1} \circ \left(q_+(M) + q_-(M) \circ \text{Ad}(M^{-1}) \right). \quad (3.20)$$

Note that the matrix $r^{\alpha\beta}(M)$ of $r(M)$ is antisymmetric. Later we shall also use the operators $r_{\pm}(M) := r(M) \pm \frac{1}{2}I$ and the corresponding $\mathcal{G} \otimes \mathcal{G}$ -valued functions on \check{G} :

$$\hat{r}(M) := r^{\alpha\beta}(M) T_{\alpha} \otimes T_{\beta}, \quad \hat{r}_{\pm}(M) = \hat{r}(M) \pm \frac{1}{2} \hat{I}, \quad \hat{I} = T^{\alpha} \otimes T_{\alpha}. \quad (3.21)$$

The above explicit description of the hamiltonian map $F \mapsto Y^F$ induced by $\kappa\Omega_{chir}^{\rho}$ on $\check{\mathcal{M}}_{chir}$ is one of our main results [21]. Its proof can be sketched as follows. First, by assuming that (3.15) holds we see from (3.12) for $Y = Y^F$ that at every point in the phase space $X(F)$ has the form (3.16) with

$$g^{-1}(x) A^F(x) g(x) = \eta(x) + q_-(M)(M^{-1} Y^F(M)), \quad (3.22)$$

$$a^F = (q_-(M) - \text{Ad } M \circ q_+(M))(M^{-1} Y^F(M)). \quad (3.23)$$

Since by the meaning of tangent vectors we must have $\eta'(x) = g^{-1}(x) Y_J^F(x) g(x)$, by taking the derivative of (3.22) we immediately get that

$$Y^F(J(x)) = [A^F(x), J(x)] + \kappa \partial_x A^F(x) \quad (3.24)$$

must hold, which in particular means that the right hand side must define a smooth, 2π -periodic function on \mathbf{R} . As for equation (3.18), this is a direct consequence of (3.22) and (3.23) by taking into account that as a tangent vector Y^F satisfies

$$M^{-1}Y^F(M) = \eta(2\pi) - M^{-1}\eta(0)M. \quad (3.25)$$

This proves that the elements of \mathbf{H} indeed meet the conditions (3.16), (3.17), (3.18). Moreover, if the hamiltonian vector field exists then by combining (3.22) and (3.23) we obtain

$$g^{-1}(x)Y^F(g(x)) = g^{-1}(x)A^F(x)g(x) - q_-(M) \circ (q_-(M) - \text{Ad } M \circ q_+(M))^{-1}(a^F), \quad (3.26)$$

which is equivalent to (3.19), since for the operator $r(M)$ defined by (3.20)

$$r_-(M) = -q_-(M) \circ (q_-(M) - \text{Ad } M \circ q_+(M))^{-1} \quad (3.27)$$

is an identity. To complete the proof, one checks that if the expression in (3.17) is 2π -periodic, then (3.19) gives a well-defined vector field (since $g^{-1}(x)Y^F(g(x+2\pi))M^{-1} - g^{-1}(x)Y^F(g(x))$ is independent of x), which satisfies (3.15) if (3.16) and (3.18) hold.

Now we elaborate the hamiltonian vector field for some particular elements in \mathbf{H} . First note that the matrix elements of the evaluation functions \mathcal{F}^x and F^x fail to satisfy the first condition, thus they are not in \mathbf{H} . However, their smeared out versions

$$\mathcal{F}_\mu := \int_0^{2\pi} dx \text{Tr}(\mu(x)J(x)), \quad F_\phi[g] := \int_0^{2\pi} dx \text{Tr}(\phi(x)g^\Lambda(x)), \quad (3.28)$$

(where in defining F_ϕ we use a representation[†] $\Lambda : G \rightarrow GL(V)$ of G with $g^\Lambda = \Lambda(g)$ and a smooth test function $\phi : \mathbf{R} \rightarrow \text{End}(V)$) can be shown to belong to \mathbf{H} , if $\mu(x)$ is a \mathcal{G} -valued, smooth, 2π -periodic test function, and ϕ satisfies $\phi^{(k)}(0) = \phi^{(k)}(2\pi) = 0$ for every integer $k \geq 0$. The corresponding hamiltonian vector fields obtained from (3.19) satisfy

$$Y^{\mathcal{F}_\mu}(g(x)) = \mu(x)g(x), \quad Y^{\mathcal{F}_\mu}(J(x)) = [\mu(x), J(x)] + \kappa\mu'(x), \quad Y^{\mathcal{F}_\mu}(M) = 0, \quad (3.29)$$

and, for $x \in [0, 2\pi]$,

$$g^{-1}(x)Y^{F_\phi}(g(x)) = \frac{1}{\kappa}T^\alpha \int_x^{2\pi} dy \text{Tr} \left(T_\alpha^\Lambda \phi(y)g^\Lambda(y) \right) - \frac{1}{2}a^{F_\phi} + r(M)(a^{F_\phi}). \quad (3.30)$$

Eq. (3.29) shows that the \mathcal{F}_μ generate an infinitesimal action of the loop group on the phase space with respect to which $g(x)$ is an affine Kac-Moody primary field, and the current $J(x)$ transforms according to the co-adjoint action of the centrally extended loop

[†]We also use the notation $\text{Tr} = c_\Lambda \text{tr}_\Lambda$, where tr_Λ is the trace over the representation Λ and c_Λ is a normalization factor that makes $c_\Lambda \text{tr}(A^\Lambda B^\Lambda)$ independent of Λ for $A, B \in \mathcal{G}$.

group. The matrix elements M_{kl}^Λ of the monodromy matrix in representation Λ also belong to \mathcal{H} . The action of $Y^{M_{kl}^\Lambda}$ on $g_{ij}^\Lambda(x)$ and on M_{ij}^Λ can be written in tensorial form as

$$Y^{M_{kl}^\Lambda}(g_{ij}^\Lambda(x)) = \frac{1}{\kappa}(g(x) \otimes M \hat{\Theta}(M))_{ik,jl}^\Lambda, \quad (3.31)$$

$$Y^{M_{kl}^\Lambda}(M_{ij}^\Lambda) = \frac{1}{\kappa}((M \otimes M) \hat{\Delta}(M))_{ik,jl}^\Lambda, \quad (3.32)$$

where our tensor product notation is $(K \otimes L)_{ik,jl} = K_{ij}L_{kl}$, and

$$\hat{\Theta}(M) = \hat{r}_+(M) - M_2^{-1}\hat{r}_-(M)M_2, \quad \hat{\Delta}(M) = \hat{\Theta}(M) - M_1^{-1}\hat{\Theta}(M)M_1 \quad (3.33)$$

with $M_1 = M \otimes 1$, $M_2 = 1 \otimes M$.

We now wish to rewrite the above hamiltonian vector fields in a symbolic notation of Poisson brackets. Recall that the PB of two smooth functions F_1 and F_2 on a *finite dimensional* smooth symplectic manifold is defined by

$$\{F_1, F_2\} = Y^{F_2}(F_1) = -Y^{F_1}(F_2) = \Omega(Y^{F_2}, Y^{F_1}), \quad (3.34)$$

where Y^{F_i} is the hamiltonian vector field associated with F_i by the symplectic form Ω . One may formally apply the same formula in the infinite dimensional case to the ‘smooth enough’ admissible functions. However, it is a non-trivial problem to precisely specify the set of functions that form a closed Poisson algebra. Setting this question aside, it is clear from (3.29) and (3.32) that the admissible functions of J and those of M will form two closed Poisson subalgebras that centralize each other. Furthermore, we may use the perfectly well-defined expression

$$\{F_\chi, F_\phi\} := Y^{F_\phi}(F_\chi) \quad (3.35)$$

for the PB of two admissible Hamiltonians of type F in eq. (3.28) to define the (‘distribution valued’) PB of the evaluation functions $g(x)$ by the equality:

$$\{F_\chi, F_\phi\} := \int_0^{2\pi} \int_0^{2\pi} dx dy \text{Tr}_{12} \left(\chi(x) \otimes \phi(y) \{g^\Lambda(x) \otimes g^\Lambda(y)\} \right), \quad (3.36)$$

where Tr_{12} is the (normalized) trace over $V \otimes V$ and $\{g^\Lambda(x) \otimes g^\Lambda(y)\}_{ik,jl} = \{g_{ij}^\Lambda(x), g_{kl}^\Lambda(y)\}$. With these definitions, our explicit formula of the hamiltonian vector field Y^{F_ϕ} in (3.30) is equivalent to the following quadratic ‘exchange algebra’ type PB for the chiral field $g(x)$:

$$\{g^\Lambda(x) \otimes g^\Lambda(y)\} = \frac{1}{\kappa} \left(g^\Lambda(x) \otimes g^\Lambda(y) \right) \left(\hat{r}(M) + \frac{1}{2} \hat{I} \text{sign}(y-x) \right)^\Lambda, \quad 0 < x, y < 2\pi. \quad (3.37)$$

Proceeding in the same way with the $\{F_\phi, M_{kl}^\Lambda\}$ PB as we did with the $\{F_\chi, F_\phi\}$ one, we conclude that the right hand side of (3.31) should be interpreted as the expression of the $\{g_{ij}^\Lambda(x), M_{kl}^\Lambda\}$ PB, and similarly for (3.32).

It is an open question if the admissible Hamiltonians of type \mathcal{F}_μ , F_ϕ and M_{kl}^Λ together generate a closed Poisson algebra. Leaving this for a future study, we here only remark that the Jacobi identity for three functions of type F_ϕ is in fact equivalent to the following equation for $\hat{r}(M)$:

$$[\hat{r}_{12}(M), \hat{r}_{23}(M)] + \Theta_{\alpha\beta}(M) T_1^\alpha \mathcal{R}^\beta \hat{r}_{23}(M) + \text{cycl. perm.} = -\frac{1}{4} \hat{f}, \quad (3.38)$$

where \hat{f} is defined by

$$\hat{f} := f_{\alpha\beta}^\gamma T^\alpha \otimes T^\beta \otimes T_\gamma, \quad [T_\alpha, T_\beta] = f_{\alpha\beta}^\gamma T_\gamma, \quad (3.39)$$

and the cyclic permutation is over the three tensorial factors with $\hat{r}_{23} = r^{\alpha\beta}(1 \otimes T_\alpha \otimes T_\beta)$, $T_1^\alpha = T^\alpha \otimes 1 \otimes 1$ and so on. We use the components of $\hat{\Theta} = \Theta_{\alpha\beta} T^\alpha \otimes T^\beta$ given by (3.33), and the left-invariant differential operators \mathcal{R}^β that act on a function ψ of M by

$$(\mathcal{R}^\beta \psi)(M) := \frac{d}{dt} \psi(M e^{t T^\beta}) \Big|_{t=0}. \quad (3.40)$$

Eq. (3.38) can be viewed as a dynamical generalization of the classical modified Yang-Baxter equation, to which it reduces if the r-matrix is a monodromy independent constant. As a consequence of $d\Omega_{chir}^\rho = 0$, (3.38) is satisfied for any $\hat{r}(M)$ given by (3.20).

What are the Poisson-Lie (PL) symmetries of the chiral WZNW phase space? To make this question more definite, we equip the group $G = \{h\}$ with a PL structure by means of the Sklyanin bracket

$$\{h \otimes h\}_{\hat{R}} = \frac{1}{\kappa} [h \otimes h, \hat{R}], \quad (3.41)$$

where $\hat{R} = R^{\alpha\beta} T_\alpha \otimes T_\beta \in \mathcal{G} \wedge \mathcal{G}$ is a *constant* r-matrix satisfying

$$[\hat{R}_{12}, \hat{R}_{23}] + \text{cycl. perm.} = -\nu^2 \hat{f} \quad (3.42)$$

for some constant ν . We then seek the conditions on $\hat{r}(M)$ and \hat{R} that guarantee the standard right action[‡] of G on $\check{\mathcal{M}}_{chir}$,

$$\check{\mathcal{M}}_{chir} \times G \ni (g, h) \mapsto gh \in \check{\mathcal{M}}_{chir}, \quad (3.43)$$

to be a PL action. This leads to the requirement

$$\hat{r}(h^{-1} M h) - \hat{R} = (h \otimes h)^{-1} (\hat{r}(M) - \hat{R}) (h \otimes h), \quad (3.44)$$

i.e., right multiplication is a PL symmetry iff the exchange r-matrix $\hat{r}(M)$ is such a solution of (3.38) that the difference $(\hat{r}(M) - \hat{R})$ is equivariant. We can provide such

[‡]Since $M \mapsto h^{-1} M h$, we here have to assume that $\check{G} \subset G$ is invariant under the adjoint action of G , or should restrict our attention to the corresponding \mathcal{G} -action.

solutions explicitly in association with *any* given solution of (3.42). These solutions are obtained by assuming the validity of the exponential parametrization for $M \in \check{G}$:

$$M := e^{2\pi\Gamma} \quad \text{for } \Gamma \in \check{\mathcal{G}} \subset \mathcal{G} \quad \text{and} \quad \mathcal{Y} := 2\pi(\text{ad}\Gamma). \quad (3.45)$$

Any analytic function of \mathcal{Y} is equivariant, and it is possible to prove [21] that the r-matrix corresponding to the linear operator

$$r(M) = \frac{1}{2} \coth \frac{\mathcal{Y}}{2} - \nu \coth(\nu \mathcal{Y}) + R \quad (3.46)$$

solves both (3.44) and (3.38) (on the domain where its power series converges).

We end this section with some remarks on the above formula. First, note that for $\nu = 0$ (3.46) is understood as the limit of the corresponding complex analytic function. Thus for $\nu = 0$ and $R = 0$ it yields $r_0 = \frac{1}{2} \coth \frac{\mathcal{Y}}{2} - \frac{1}{\mathcal{Y}}$. If the PB (3.37) on $\check{\mathcal{M}}_{chir}$ is defined by r_0 , then (3.43) is a *classical G-symmetry*. Second, if $\nu = \frac{1}{2}$ then $r = R$, which is the case of the constant exchange r-matrices [15]. Third, it is worth stressing that for a compact Lie algebra \mathcal{G} constant exchange r-matrices do not exist, because of the negative sign on the right hand side of (3.38), but our formula (3.46) gives explicit solutions of (3.38) also in this case using a purely imaginary ν in (3.42). Finally, we remark that in the $\nu = \frac{1}{2}$ case the construction of the 2-form ρ that corresponds to the r-matrix in (3.46) is presented in [15], while in general a suitable local 2-form can be obtained by solving (3.20) for q . Further comments are contained in [21].

4 Hamiltonian structures for diagonal monodromy

We now describe the hamiltonian structure that results by restricting the symplectic form Ω_{chir}^ρ to a submanifold $\mathcal{M}_{Bloch} \subset \mathcal{M}_{chir}$ consisting of chiral WZNW fields with diagonal monodromy. The corresponding exchange algebra PB turns out to contain the classical dynamical r-matrix (4.32).

In this section, let \mathcal{G} be either a complex simple Lie algebra or its normal real form, and G a corresponding Lie group. Choose a Cartan subalgebra $\mathcal{H} \subset \mathcal{G}$ that admits the root space decomposition

$$\mathcal{G} = \mathcal{H} \oplus \sum_{\alpha \in \Phi} \mathcal{G}_\alpha, \quad (4.1)$$

and an associated basis $H_k \in \mathcal{H}$, $E_\alpha \in \mathcal{G}_\alpha$ normalized by $\text{Tr}(E_\alpha E_{-\alpha}) = \frac{2}{|\alpha|^2}$. By using this basis any $A \in \mathcal{G}$ can be decomposed as

$$A = A^0 + A^r \quad \text{with} \quad A^0 \in \mathcal{H}, \quad A^r = \sum_{\alpha \in \Phi} E_\alpha \text{Tr}(E^\alpha A), \quad E^\alpha := \frac{1}{2} |\alpha|^2 E_{-\alpha}. \quad (4.2)$$

Fix an open domain $\mathcal{A} \subset \mathcal{H}$ which has the properties that $\alpha(\omega) \notin i2\pi\mathbf{Z}$ for any root, $\alpha \in \Phi \subset \mathcal{H}^*$, and the map $\mathcal{A} \ni \omega \mapsto e^\omega \in G$ is injective.

Then define $\mathcal{M}_{Bloch} \subset \mathcal{M}_{chir}$ by

$$\mathcal{M}_{Bloch} := \{b \in C^\infty(\mathbf{R}, G) \mid b(x+2\pi) = b(x)e^\omega, \quad \omega \in \mathcal{A} \subset \mathcal{H}\}. \quad (4.3)$$

Let \mathcal{M}_{Bloch} be equipped with the 2-form $\kappa\Omega_{Bloch}^{\rho_B}$, where

$$\Omega_{Bloch}^{\rho_B}(b) := -\frac{1}{2} \int_0^{2\pi} dx \operatorname{Tr} \left(b^{-1} db \right) \wedge \left(b^{-1} db \right)' - \frac{1}{2} \operatorname{Tr} \left((b^{-1} db)(0) \wedge d\omega \right) + \rho_B(\omega) \quad (4.4)$$

with an arbitrary *closed* 2-form ρ_B on \mathcal{A} . Clearly, $\Omega_{Bloch}^{\rho_B}$ could be obtained from Ω_{chir}^ρ (2.21) upon imposing the constraint $M = e^\omega$. Now ρ_B is parametrized as

$$\rho_B(\omega) = \frac{1}{2} q_B^{kl}(\omega) \operatorname{Tr} (H_k d\omega) \wedge \operatorname{Tr} (H_l d\omega), \quad q_B^{kl} = -q_B^{lk}, \quad (4.5)$$

and a corresponding linear operator $q_B(\omega)$ on \mathcal{H} is defined by

$$q_B(\omega)(C) = H_k q_B^{kl}(\omega) \operatorname{Tr} (H_l C) \quad \forall C \in \mathcal{H}. \quad (4.6)$$

To show that $\Omega_{Bloch}^{\rho_B}$ is symplectic, it will be convenient to parametrize $b \in \mathcal{M}_{Bloch}$ as

$$b(x) = h(x) \exp(x\bar{\omega}), \quad \bar{\omega} := \frac{\omega}{2\pi}, \quad (4.7)$$

where $\omega \in \mathcal{A}$ and $h \in \tilde{G}$. This one-to-one parametrization yields the identification

$$\mathcal{M}_{Bloch} = \tilde{G} \times \mathcal{A} = \{(h, \omega)\}. \quad (4.8)$$

Correspondingly, a vector field X on \mathcal{M}_{Bloch} is parametrized by

$$X = (X_h, X_\omega) \quad X_h \in T_h \tilde{G} \quad X_\omega \in T_\omega \mathcal{A} \simeq \mathcal{H} \quad (4.9)$$

with $h^{-1}X_h \in T_e \tilde{G} \simeq \tilde{\mathcal{G}}$. By regarding ω and h as evaluation functions on \mathcal{M}_{Bloch} , we may write $X_\omega = X(\omega)$ and $X_h(x) = X(h(x))$. Equivalently, X can be characterized by its action on $b(x)$,

$$b^{-1}(x)X(b(x)) = e^{-x\bar{\omega}}h^{-1}(x)X(h(x))e^{x\bar{\omega}} + xX(\bar{\omega}), \quad (4.10)$$

where the function $b^{-1}(x)X(b(x))$ on \mathbf{R} is uniquely determined by its restriction to $[0, 2\pi]$. Naturally, the derivative $X(F)$ of a function F on \mathcal{M}_{Bloch} is defined by using that any vector is the velocity to a smooth curve. That is, if the value of the vector field X at $b \in \mathcal{M}_{Bloch}$ coincides with the velocity to the curve $\gamma(x, t)$ at $t = 0$, $\gamma(x, 0) = b(x)$, then for a differentiable function F we have $X(F)[b] = \frac{d}{dt}F[\gamma(x, t)]|_{t=0}$.

Arguing similarly to section 3, it can be shown that $\Omega_{Bloch}^{\rho_B}$ is weakly non-degenerate on \mathcal{M}_{Bloch} for any ρ_B . The *admissible* Hamiltonians that possess hamiltonian vector fields now turn out to be those functions F on \mathcal{M}_{Bloch} whose derivative with respect to any vector field X exists and has the form

$$X(F) = \langle dF, X \rangle = \operatorname{Tr} (d_\omega F X_\omega) + \int_0^{2\pi} dx \operatorname{Tr} \left((h^{-1}d_h F)(h^{-1}X_h) \right) \quad (4.11)$$

where

$$dF = (d_h F, d_\omega F) \quad \text{with} \quad d_h F \in T_h^* \tilde{G}, \quad d_\omega F \in T_\omega^* \mathcal{A} \quad (4.12)$$

is the exterior derivative of F . We here identify $T_\omega^* \mathcal{A}$ with \mathcal{H} by means of the scalar product Tr and also identify $T_e^* \tilde{G}$ with $\tilde{\mathcal{G}}$ by the scalar product $\int_0^{2\pi} \text{Tr}(\cdot, \cdot)$, whereby we have $h^{-1} d_h F \in T_e^* \tilde{G} = \tilde{\mathcal{G}}$. It is clear that the local evaluation functions $\mathcal{M}_{Bloch} \ni b \mapsto b_{kl}^\Lambda(x)$ (with the matrix elements b_{kl}^Λ taken in some representation Λ of G) are differentiable but not admissible, while e.g. the Fourier coefficients of the components of $J = \kappa b' b^{-1}$ as well as the components of ω yield admissible Hamiltonians.

Next we prove that $\kappa \Omega_{Bloch}^{\rho_B}$ indeed permits to associate a unique hamiltonian vector field, Y^F , with any Hamiltonian, F , subject to (4.11), (4.12). By definition, Y^F must satisfy

$$\langle dF, X \rangle = X(F) = \kappa \Omega_{Bloch}^{\rho_B}(X, Y^F) \quad (4.13)$$

for any vector field X . To determine Y^F , we first point out that in terms of (h, ω)

$$\begin{aligned} \Omega_{Bloch}^{\rho_B}(h, \omega) &= -\frac{1}{2} \int_0^{2\pi} dx \text{Tr} \left((h^{-1} dh) \wedge (h^{-1} dh)' \right. \\ &\quad \left. + 2\bar{\omega} (h^{-1} dh) \wedge (h^{-1} dh) - 2d\bar{\omega} \wedge h^{-1} dh \right) + \rho_B(\omega). \end{aligned} \quad (4.14)$$

By equating the coefficients of $h^{-1} X(h)$ and $X(\omega)$ on the two sides of (4.13), we obtain the following equations for Y^F :

$$2\pi \left(h^{-1} Y^F(h) \right)' + [h^{-1} Y^F(h), \omega] + Y^F(\omega) = -\frac{2\pi}{\kappa} h^{-1} d_h F \quad (4.15)$$

$$2\pi q_B(\omega)(Y^F(\omega)) + \int_0^{2\pi} dx (h^{-1} Y^F(h))^0 = \frac{2\pi}{\kappa} d_\omega F. \quad (4.16)$$

Given $d_h F$ and $d_\omega F$, we will determine $b^{-1} Y^F(b)$, which is equivalent to finding $h^{-1} Y^F(h)$ and $Y^F(\omega)$.

On account of (4.10), (4.15) is in fact equivalent to

$$\left(b^{-1} Y^F(b) \right)'(x) = -\frac{1}{\kappa} e^{-\bar{\omega}x} (h^{-1} d_h F)(x) e^{\bar{\omega}x}, \quad (4.17)$$

whose solution is

$$b^{-1}(x) Y^F(b(x)) = b^{-1}(0) Y^F(b(0)) - \frac{1}{\kappa} \int_0^x dy e^{-\bar{\omega}y} (h^{-1} d_h F)(y) e^{\bar{\omega}y}. \quad (4.18)$$

Hence the only non-trivial problem is to determine the initial value

$$Q_F := b^{-1}(0) Y^F(b(0)) = h^{-1}(0) Y^F(h(0)). \quad (4.19)$$

To this end, note from (4.10) that

$$Y^F(\omega) = e^\omega b^{-1}(2\pi) Y^F(b(2\pi)) e^{-\omega} - b^{-1}(0) Y^F(b(0)). \quad (4.20)$$

By using (4.18), the Cartan part of (4.20) requires that

$$Y^F(\omega) = -\frac{1}{\kappa} \int_0^{2\pi} dx (h^{-1} d_h F)^0(x), \quad (4.21)$$

while the root part of (4.20) gives

$$e^{-\omega} Q_F^r e^{\omega} - Q_F^r = -\frac{1}{\kappa} \int_0^{2\pi} dx e^{-\bar{\omega}x} (h^{-1} d_h F)^r(x) e^{\bar{\omega}x}, \quad (4.22)$$

where $Q_F = Q_F^0 + Q_F^r$ according to (4.2). Then (4.22) completely determines Q_F^r as

$$Q_F^r = \frac{1}{\kappa} \sum_{\alpha \in \Phi} \frac{E_\alpha}{1 - e^{-\alpha(\omega)}} \int_0^{2\pi} dx e^{-\alpha(\bar{\omega})x} \text{Tr} \left((h^{-1} d_h F)(x) E^\alpha \right). \quad (4.23)$$

As for the remaining unknown, Q_F^0 , (4.16) with (4.10) and (4.20) leads to the result:

$$2\pi\kappa Q_F^0 = 2\pi d_\omega F + (2\pi q_B(\omega) - \pi I) \left(\int_0^{2\pi} dx (h^{-1} d_h F)^0(x) \right) + \int_0^{2\pi} dx \int_0^x dy (h^{-1} d_h F)^0(y). \quad (4.24)$$

In conclusion, we have found that the hamiltonian vector field $b^{-1}Y^F(b)$ is uniquely determined and is explicitly given by (4.18) with $b^{-1}(0)Y^F(b(0)) = Q_F$ in (4.23), (4.24). In the derivation of Y^F we have crucially used that ω is restricted to the domain $\mathcal{A} \subset \mathcal{H}$. At the excluded points of \mathcal{H} some denominators in (4.23) may vanish, whereby $\Omega_{Bloch}^{\rho B}$ becomes singular.

The Poisson bracket of two ‘smooth enough’ admissible Hamiltonians F_1 and F_2 on \mathcal{M}_{Bloch} is determined by the formula $\{F_1, F_2\} = \kappa \Omega_{Bloch}^{\rho B}(Y^{F_2}, Y^{F_1})$ and now we extract a ‘classical exchange algebra’ from this formula. Analogously to the previous section, for this we consider functions of the form

$$F_\phi(h, \omega) = \int_0^{2\pi} dx \text{Tr}(\phi(x) b^\Lambda(x)), \quad (4.25)$$

where $b^\Lambda(x)$ is taken in a representation Λ of G and $\phi(x)$ is a smooth, matrix valued, smearing-function in that representation. It is easy to check that F_ϕ is admissible if

$$\phi^{(k)}(0) = \phi^{(k)}(2\pi) = 0 \quad \forall k = 0, 1, 2, \dots, \quad (4.26)$$

and the exterior derivative of F_ϕ at (h, ω) is given by

$$(d_\omega F_\phi)(h, \omega) = \frac{1}{2\pi} \sum_k H^k \text{Tr} \left(H_k^\Lambda \int_0^{2\pi} dx (x \phi(x) b^\Lambda(x)) \right), \quad (4.27)$$

$$\left((h^{-1} d_h F_\phi)(h, \omega) \right)(x) = \sum_a T^a \text{Tr} \left(\phi(x) h^\Lambda(x) T_a^\Lambda e^{x\bar{\omega}^\Lambda} \right) \quad \text{for } x \in [0, 2\pi]. \quad (4.28)$$

We here denote by H_k , H^k and T_a , T^a dual bases of \mathcal{H} and \mathcal{G} , respectively. The last formula extends to a smooth 2π -periodic function on the real line precisely if (4.26) is satisfied. The hamiltonian vector field Y^{F_ϕ} is then found to be

$$\left(b^{-1} Y^{F_\phi}(b) \right)(x) = Q_{F_\phi} - \frac{1}{\kappa} \sum_a T^a \int_0^x dy \text{Tr}(\phi(y) b^\Lambda(y) T_a^\Lambda), \quad \text{for } x \in [0, 2\pi], \quad (4.29)$$

where Q_{F_ϕ} is determined as described above. By combining the preceding formulae, one can verify that

$$\{F_\chi, F_\phi\} = \kappa \Omega_{Bloch}^{\rho B}(Y^{F_\phi}, Y^{F_\chi}) = \int_0^{2\pi} \int_0^{2\pi} dx dy \text{Tr}_{12} \left(\chi(x) \otimes \phi(y) \{b^\Lambda(x) \otimes b^\Lambda(y)\} \right) \quad (4.30)$$

holds for any ϕ, χ subject to (4.26) *provided that one has*

$$\{b^\Lambda(x) \otimes b^\Lambda(y)\} = \frac{1}{\kappa} \left(b^\Lambda(x) \otimes b^\Lambda(y) \right) \left(\hat{\mathcal{R}}(\omega) + \frac{1}{2} \hat{I} \text{sign}(y-x) \right)^\Lambda, \quad 0 < x, y < 2\pi \quad (4.31)$$

with the dynamical r-matrix

$$\hat{\mathcal{R}}(\omega) = \frac{1}{4} \sum_{\alpha \in \Phi} |\alpha|^2 \coth\left(\frac{1}{2}\alpha(\omega)\right) E_\alpha \otimes E_{-\alpha} + \sum_{kl} q_B^{kl}(\omega) H_k \otimes H_l. \quad (4.32)$$

The local formula (4.31) completely encodes the Poisson brackets on \mathcal{M}_{Bloch} since Y^{F_ϕ} can be recovered if the right hand side of (4.30) is given.

The hamiltonian vector fields associated with $\omega_k := \text{Tr}(\omega H_k)$ and with the functions \mathcal{F}_μ of J (see (3.28)) can be checked to be

$$Y^{\omega_k}(b(x)) = \frac{1}{\kappa} b(x) H_k, \quad Y^{\mathcal{F}_\mu}(b(x)) = \mu(x) b(x). \quad (4.33)$$

Thus J generates an action of the affine Kac-Moody algebra on \mathcal{M}_{Bloch} centralized by the action of \mathcal{H} generated by ω .

The dynamical r-matrix (4.32) is antisymmetric, and is neutral in the sense that

$$[H_k \otimes 1 + 1 \otimes H_k, \hat{\mathcal{R}}(\omega)] = 0, \quad (4.34)$$

which ensures the validity of the Jacobi identity for the three functions F_ϕ, F_χ, ω_k . Moreover, it satisfies the equation

$$[\hat{\mathcal{R}}_{12}(\omega), \hat{\mathcal{R}}_{23}(\omega)] + \sum_k H_1^k \frac{\partial}{\partial \omega^k} \hat{\mathcal{R}}_{23}(\omega) + \text{cycl. perm.} = -\frac{1}{4} \hat{f} \quad (4.35)$$

that ensures the Jacobi identity for three functions of type F_ϕ . This dynamical generalization of the modified classical Yang-Baxter equation arises in other contexts as well [25, 26, 27] and has been much studied recently [28, 29, 30]. The exchange r-matrix (4.32) of the chiral WZNW Bloch waves was first obtained in [9], where it was also shown that it satisfies (4.35).

5 Poisson-Lie groupoids from chiral WZNW

It is well-known [31] that one can associate a Poisson-Lie (PL) group with any anti-symmetric solution of the modified classical Yang-Baxter equation. Remarkably, the

dynamical generalizations of this equation that arise in the WZNW model permit analogous interpretations in terms of PL groupoids. In particular, this means that one can associate a PL groupoid with any chiral extension of the WZNW phase space. Below we briefly describe these groupoids, which are finite dimensional Poisson manifolds that encode the (non Kac-Moody aspects of the) infinite dimensional chiral WZNW PBs.

Roughly speaking, a groupoid is a set, say P , endowed with a ‘partial multiplication’ that behaves similarly to a group multiplication in the cases when it can be performed (see e.g. [32]). In the cases of our interest $P = S \times G \times S = \{(M^F, g, M^I)\}$, where G is a group and S is some set. The partial multiplication is defined for those triples (M^F, g, M^I) and $(\bar{M}^F, \bar{g}, \bar{M}^I)$ for which $M^I = \bar{M}^F$, and the product is

$$(M^F, g, M^I)(\bar{M}^F, \bar{g}, \bar{M}^I) := (M^F, g\bar{g}, \bar{M}^I) \quad \text{for } M^I = \bar{M}^F. \quad (5.1)$$

Thus the graph of the partial multiplication is the subset of

$$P \times P \times P = \{(M^F, g, M^I)\} \times \{(\bar{M}^F, \bar{g}, \bar{M}^I)\} \times \{(\hat{M}^F, \hat{g}, \hat{M}^I)\} \quad (5.2)$$

defined by the constraints

$$M^I = \bar{M}^F, \quad \hat{M}^F = M^F, \quad \hat{M}^I = \bar{M}^I, \quad \hat{g} = g\bar{g}, \quad (5.3)$$

where the hatted triple encodes the components of the product. A PL groupoid [33] is a (Lie) groupoid P , which is also a Poisson manifold in such a way that the graph of the partial multiplication is a *coisotropic* submanifold of $P \times P \times P^-$, where P^- denotes the manifold P endowed with the opposite of the PB on P . To put it differently, *the constraints that define the graph are first class*. (For $P = S \times G \times S$, this definition reduces to that of a PL group if S consists of a single point.)

Let us first recall the definition of the PL groupoids that are related to the Poisson structures on \mathcal{M}_{Bloch} . In a more general context, these groupoids have been introduced in [28]. They are of the form above, where S is a domain in the dual of a Cartan subalgebra \mathcal{H} of a simple Lie group G . We now identify \mathcal{H}^* with \mathcal{H} and take the domain to be $\mathcal{A} \subset \mathcal{H}$ considered in section 4. For notational convenience, we further identify S with $\exp(\mathcal{A}) \subset \exp(\mathcal{H})$, and denote the components of the corresponding triples as $M^I = \exp(\omega^I)$ and $M^F = \exp(\omega^F)$ for $\omega^F, \omega^I \in \mathcal{A}$. Using the standard tensorial notation and a basis H_i of \mathcal{H} , we then define a Poisson structure on $P_{Bloch} := \mathcal{A} \times G \times \mathcal{A} = \{(\omega^F, g, \omega^I)\}$ as follows:

$$\begin{aligned} \kappa\{g_1, g_2\}_{P_{Bloch}} &= g_1 g_2 \hat{\mathcal{R}}(\omega^I) - \hat{\mathcal{R}}(\omega^F) g_1 g_2 \\ \kappa\{g, \omega_i^I\}_{P_{Bloch}} &= g H_i \\ \kappa\{g, \omega_i^F\}_{P_{Bloch}} &= H_i g \\ \{\omega_i^I, \omega_j^I\}_{P_{Bloch}} &= \{\omega_i^F, \omega_j^F\}_{P_{Bloch}} = \{\omega_i^I, \omega_j^F\}_{P_{Bloch}} = 0, \end{aligned} \quad (5.4)$$

where κ is a constant included for comparison purposes and $\omega_i^{I(F)} = \text{Tr}(H_i \omega^{I(F)})$. Equations (4.34) and (4.35) for $\hat{\mathcal{R}}(\omega)$ are sufficient for the Jacobi identities of this PB to be satisfied [28]. One can also check that the graph of the partial multiplication is coisotropic.

In some sense, the PBs (5.4) on P_{Bloch} correspond to the PBs on the chiral WZNW phase space $\mathcal{M}_{\text{Bloch}}$. Motivated by this, we now define a PL groupoid which is related to an arbitrary chiral extension of the WZNW phase space with generic monodromy. In this case, using an open domain $\check{G} \subset G$, we take P to be

$$P = \check{G} \times G \times \check{G}, \quad (5.5)$$

and postulate on it a PB $\{ , \}_P$ as follows:

$$\begin{aligned} \kappa\{g_1, g_2\}_P &= g_1 g_2 \hat{r}(M^I) - \hat{r}(M^F) g_1 g_2 \\ \kappa\{g_1, M_2^I\}_P &= g_1 M_2^I \hat{\Theta}(M^I) \\ \kappa\{g_1, M_2^F\}_P &= M_2^F \hat{\Theta}(M^F) g_1 \\ \kappa\{M_1^I, M_2^I\}_P &= M_1^I M_2^I \hat{\Delta}(M^I) \\ \kappa\{M_1^F, M_2^F\}_P &= -M_1^F M_2^F \hat{\Delta}(M^F) \\ \kappa\{M_1^I, M_2^F\}_P &= 0. \end{aligned} \quad (5.6)$$

It is easy to verify that a PB given by the ansatz (5.6) always yields a PL groupoid, since the constraints in (5.3) are first class for any choice of the $\mathcal{G} \otimes \mathcal{G}$ valued ‘structure functions’ \hat{r} , $\hat{\Theta}$, $\hat{\Delta}$ on \check{G} . Of course, the structure functions must satisfy a system of equations for the ansatz (5.6) to define a PB. These equations are spelled out in [21]. The important point is that, in fact, a *sufficient condition* for the Jacobi identity is obtained by assuming that $\hat{\Theta}(M)$ and $\hat{\Delta}(M)$ are given by (3.33) in terms of an antisymmetric solution $\hat{r}(M)$ of (3.38).

We have extracted a PL groupoid from any chiral extension of the WZNW phase space with generic monodromy by taking the triple \hat{r} , $\hat{\Theta}$, $\hat{\Delta}$ that arises in the WZNW model to be the structure functions of $\{ , \}_P$. It should be noticed that if \hat{r} is non-dynamical, then the PL groupoid P carries the same information as the group G endowed with the corresponding Sklyanin bracket. Among our PL groupoids there are also those special cases for which $(\hat{r} - \hat{R})$ satisfies the equivariance condition (3.44) in relation with an arbitrary constant r-matrix \hat{R} subject to (3.42). In these cases, it is possible to define commuting left and right PL actions of the group G (endowed with the PB (3.41)) on P , reflecting the corresponding PL symmetry (3.43) on the chiral WZNW phase space.

We believe that it would be interesting to study the above introduced PL groupoids further, for example to understand their quantization and relate them to the quantized (chiral) WZNW conformal field theory.

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